On the Real Spectra of Calogero Model with Complex Coupling

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Abstract

We study the eigenvalue problem of the rational Calogero model with the coupling of the inverse-square interaction as a complex number. We show that although this model is manifestly non-invariant under the combined parity and time-reversal symmetry $\mathcal{PT}$, the eigenstates corresponding to the zero value of the generalized angular momentum have real energies.

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The standard practice in quantum mechanics is to consider self-adjoint operators so that the corresponding spectrum is real and consequently, the time-evolution of the states is unitary. However, it has been found recently that a non-Hermitian Hamiltonian invariant under the combined $\mathcal{PT}$ symmetry can have real spectrum [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. The spectrum is entirely real if the $\mathcal{PT}$ symmetry is unbroken. On the other hand, part of the spectrum is real if the $\mathcal{PT}$ symmetry is broken spontaneously[1].

In this Letter we study the rational Calogero model [11, 12] with complex coupling for the many-body interaction. Several extensions of the Calogero model with non-Hermitian Hamiltonians have been studied in the literature [9, 10]. However, all these models respect $\mathcal{PT}$ symmetry and lead to real spectrum. The Hamiltonian we study is identical to that of the original N-body rational Calogero model except that the coefficient of the inverse square interaction is complex. This model is non-Hermitian, manifestly non-invariant under $\mathcal{PT}$, but nevertheless is shown to admit a partly real spectrum, with the ground state energy always real.

Following the analysis of Calogero [11], the eigenvalue equation of the rational Calogero model with complex coupling for the many-body inverse-square interaction can be reduced to the eigenvalue equation of an effective single particle Hamiltonian containing harmonic and inverse-square interaction. We show that for certain values of the complex coupling of the inverse square interaction, this effective Hamiltonian can indeed be Hermitian when the generalized angular momentum [11] is zero. This however happens only for the strongly attractive values of the inverse square coupling term in the effective Hamiltonian. Such a strongly attractive single particle system in absence of the harmonic term has already been analyzed by Case [13], who obtained the corresponding spectrum in terms of an undetermined parameter which has the physical interpretation of a cutoff in the coordinate space. In our analysis, we explicitly introduce a cutoff which prevents the particles to occupy the same position simultaneously, i.e. it serves as a cutoff in the coordinate space. In the analysis of ref. [13], the spectrum of the strongly attractive model in absence of the harmonic term is unbounded from below. In the model under consideration here, which includes the harmonic term, the spectrum is however bounded from below for finite nonzero values of the cutoff.

The Hamiltonian of the rational Calogero model is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \left[\frac{a^2 - \frac{1}{4}}{(x_i - x_j)^2} + \frac{\Omega^2}{16}(x_i - x_j)^2 \right] \tag{1}$$

where a and Ω are constants, x_i is the coordinate of the ith particle and units have been chosen such that $2m\hbar^{-2} = 1$. The constant a is conventionally taken to be real, which is required for the operator H to be Hermitian. In this Letter we shall however show that for certain complex values of a, the eigenvalue problem

$$H\Psi = E\Psi \tag{2}$$

still admits a real spectrum with normalizable solutions. In the discussion below, we therefore take $a = a_R + ia_I$ where a_R and a_I are the real and imaginary parts of a. The Hamiltonian H is invariant under parity $x_i \to -x_i$, while non-invariant under the time-reversal symmetry $i \to -i$. Thus, the combined $\mathcal{PT}$ symmetry is not respected by H.

Following [11], we consider the above eigenvalue equation in a sector of configuration space corresponding to a definite ordering of particles given by $x_1 \geq x_2 \geq \cdots \geq x_N$. The translation-invariant eigenfunctions of the Hamiltonian H can be written as

$$\Psi = \prod_{i<j} (x_i - x_j)^{a+\frac{1}{2}} \ \phi(r) \ P_k(x), \tag{3}$$

where $x \equiv (x_1, x_2, \ldots, x_N)$,

$$r^2 = \frac{1}{N} \sum_{i<j} (x_i - x_j)^2 \tag{4}$$

and $P_k(x)$ is a translation-invariant as well as homogeneous polynomial of degree $k(\geq 0)$ which satisfies the equation

$$\left[\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{2(a + \frac{1}{2})}{(x_i - x_j)} \frac{\partial}{\partial x_i} \right] P_k(x) = 0. \tag{5}$$

The existence of complete solutions of (5) for real a has been discussed by Calogero [11]. For both real and complex a, $P_0(x)$ is a constant and is a solution of Eqn. (5). For $k \neq 0$, the solutions of Eqn. (5) for complex a are obtained by analytic continuation in the parameter space a.

Substituting Eqn. (3) in Eqn. (2), using Eqns. (4- 5) and making a further substitution $\phi = r^{-(\mu+\frac{1}{2})}\Phi$ we get

$$\tilde{H}\Phi = E\Phi, \tag{6}$$

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where

$$\tilde{H} = -\frac{d^2}{dr^2} + \frac{(\mu^2 - \frac{1}{4})}{r^2} + \omega^2 r^2, \tag{7}$$

with $\omega^2 = \frac{1}{8}\Omega^2 N$ and

$$\mu = k + \frac{1}{2}(N-3) + \frac{1}{2}N(N-1)(a_R + ia_I + \frac{1}{2}). \tag{8}$$

The operator $\tilde{H}$ can be interpreted as the effective Hamiltonian of a particle in the combined harmonic plus inverse-square interaction. Let us now choose the real part of a as

$$a_R = -\frac{N-3}{N(N-1)} - \frac{1}{2}, \tag{9}$$

and keep a_I as an arbitrary real number. It may be noted that the wavefunction Ψ is square-integrable for $a_R + \frac{1}{2} > -\frac{1}{2}$ [14, 15], which is indeed satisfied for the above choice of a_R for $N \geq 3$. With the above choice of the complex coupling a, for $k = 0$ we see that

$$\mu = \frac{i}{2}N(N-1)a_I \equiv i\nu \tag{10}$$

is purely imaginary and the operator $\tilde{H}$ takes the form

$$\tilde{H} = -\frac{d^2}{dr^2} - \frac{(\nu^2 + \frac{1}{4})}{r^2} + \omega^2 r^2, \tag{11}$$

which is Hermitian and is expected to have real eigenvalues. However, for the same choice of a_R, when $k \neq 0$, μ is in general a complex quantity. Thus the eigenvalues of $\tilde{H}$ in the $k \neq 0$ sector in general would not be real. Below we shall first find the eigenvalues of the operator $\tilde{H}$ for $k = 0$ and then provide a suitable interpretation of the states in the $k \neq 0$ sector.

We now proceed to solve the eigenvalue problem given in Eqn. (6) with $\tilde{H}$ given by Eqn. (11). In this case, the inverse-square interaction is necessarily in the strongly attractive regime for $\nu \neq 0$. As mentioned before, we put a cutoff r_0 in the coordinate space which leads to the boundary condition $\Phi(r) = 0$ for $r = r_0$. We also demand that $\Phi(r) \in L^2(R^+)$. Defining $q = \omega r^2$ and $\Phi = q^{-\frac{1}{4}}\chi(q)$, the eigenvalue equation $\tilde{H}\Phi = E\Phi$ can be written as,

$$\frac{d^2\chi}{dq^2} + \left[-\frac{1}{4} + \frac{1}{4q^2}(1+\nu^2) + \frac{E}{4wq}\right]\chi = 0, \quad \chi(q_0 = wr_0^2) = 0. \tag{12}$$

The above equation can be identified as the Whittaker's equation, the two linearly independent solutions of which are given by

$$W_{\frac{E}{4\omega}, \pm\frac{i\nu}{2}}(q) = e^{-\frac{q}{2}}q^{\frac{1\pm i\nu}{2}} M\left(\frac{1\pm i\nu}{2} - \frac{E}{4\omega}, 1 \pm i\nu, q\right), \tag{13}$$

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where $M\left(\frac{1\pm i\nu}{2} - \frac{E}{4\omega}, 1\pm i\nu, q\right)$ is Kummer's function [16]. The general solution of Eqn. (12) satisfying the boundary condition at q_0 is given by

$$\chi(q) = A\left[W_{\frac{E}{4\omega},\frac{i\nu}{2}}(q)\, W_{\frac{E}{4\omega},-\frac{i\nu}{2}}(q_0) - W_{\frac{E}{4\omega},-\frac{i\nu}{2}}(q)\, W_{\frac{E}{4\omega},\frac{i\nu}{2}}(q_0)\right], \tag{14}$$

where A is the normalization constant. In the limit $q \to \infty$, we have [16]

$$\frac{M(a,b,q)}{\Gamma(b)} \equiv \frac{e^{i\pi a}q^{-a}}{\Gamma(b-a)} + \frac{e^q q^{a-b}}{\Gamma(a)} + O\left(\frac{1}{q}\right). \tag{15}$$

Thus, as $q \to \infty$,

$$\chi(q) \to Ae^{\frac{q}{2}}\left[\frac{\Gamma(1+i\nu)}{\Gamma(\frac{1+i\nu}{2} - \frac{E}{4\omega})}W_{\frac{E}{4\omega},-\frac{i\nu}{2}}(q_0) - \frac{\Gamma(1-i\nu)}{\Gamma(\frac{1-i\nu}{2} - \frac{E}{4\omega})}W_{\frac{E}{4\omega},\frac{i\nu}{2}}(q_0)\right]. \tag{16}$$

The square integrability of χ can thus be ensured if the quantity in the parenthesis in Eqn. (16) is identically zero, i.e.

$$\frac{W_{\frac{E}{4\omega},-\frac{i\nu}{2}}(q_0)}{W_{\frac{E}{4\omega},\frac{i\nu}{2}}(q_0)} = \frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)}\frac{\Gamma(\frac{1+i\nu}{2} - \frac{E}{4\omega})}{\Gamma(\frac{1-i\nu}{2} - \frac{E}{4\omega})}. \tag{17}$$

We are interested in the solution of the eigenvalue equation in the limit of a small cutoff, i.e. when $q_0 \to 0$. In this limit, we have $W_{\frac{E}{4\omega},-\frac{i\nu}{2}}(q_0) \to q_0^{\frac{1-i\nu}{2}}$ [16]. The energy eigenvalues are thus determined from the equation,

$$q_0^{-i\nu} = \frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)}\frac{\Gamma(\frac{1+i\nu}{2} - \frac{E}{4\omega})}{\Gamma(\frac{1-i\nu}{2} - \frac{E}{4\omega})}, \tag{18}$$

or equivalently from

$$e^{-i(\nu \ln q_0 + 2\theta)} = e^{2i\alpha}, \tag{19}$$

where θ and α are the arguments in the polar representation of $\Gamma(1-i\nu)$ and $\Gamma(\frac{1+i\nu}{2} - \frac{E}{4\omega})$ respectively. The spectrum is obtained by solving Eqn. (19) graphically. We have studied Eqn. (19) using Mathematica for different values of ν and q_0. For a fixed ν and q_0, there is one negative energy bound state and infinitely many positive energy bound states. Moreover, the energy levels are not equispaced. The absolute value of the negative energy depends on the choice of q_0. For smaller values for q_0, $|E|$ for the negative energy eigenstate increases. We plot our results in Fig.1 and Fig. 2.

The following points about the spectrum in the $k = 0$ sector may be noted:

1) As shown above, even for certain complex values of the inverse-square coupling, the rational Calogero model admits a real spectrum in the $k = 0$ sector. The real part of the
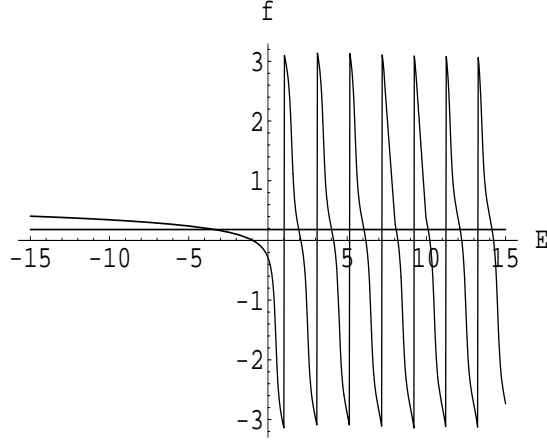
FIG. 1: A plot of Eqn. (19) with $w = 0.25$, $\nu = 0.3$ and $q_0 = 0.1$. The horizontal straight line corresponds to the value of the l.h.s of Eqn. (19) and $f = 2\alpha$.
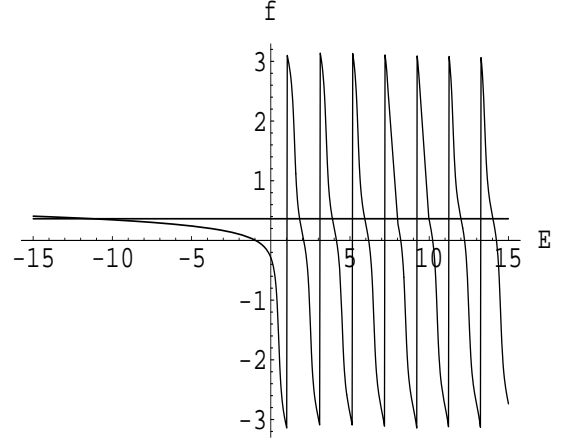
FIG. 2: A plot of Eqn. (19) with $w = 0.25$, $\nu = 0.3$ and $q_0 = .03$. The negative energy state has a lower eigenvalue due to the choice of a smaller value of the cutoff q_0.

relevant coupling is N dependent and is given by Eqn. (9) whereas the imaginary part of the coupling is arbitrary. Thus there is a range of values of the parameter a for which the spectrum in the $k = 0$ sector is real.

2) The spectrum here for a given value of q_0 and ν is bounded from below. In order to see this, consider $E = -\mathcal{E}$ with $\mathcal{E} \to \infty$. In this limit, the argument of the $\Gamma(\frac{1+i\nu}{2} - \frac{E}{4\omega})$ can be approximated by $\frac{\nu}{2}\psi(\frac{1}{2} + \frac{\mathcal{E}}{4w})$ and Eqn. (18) can be written as

$$-\nu \ln q_0 - 2\theta = \nu\psi\left(\frac{1}{2} + \frac{\mathcal{E}}{4w}\right), \tag{20}$$

where ψ denotes the digamma function [16]. For positive values of its argument, the digamma function is monotonically increasing. Thus, for any given finite but nonzero value of q_0 and ν, $E \to -\infty$ is not a part of the spectrum.

3) This system generically admits a single negative energy eigenstate. Existence of negative energy eigenstates in the Calogero model is associated with the self-adjoint extension of the corresponding Hamiltonian [14, 15]. We have however not studied the self-adjointness of the effective Hamiltonian $\tilde{H}$ here.

As mentioned before, the parameter μ is complex when $k \neq 0$. In that case the effective Hamiltonian would not be Hermitian and the states with $k \neq 0$ are expected to have complex eigenvalues. These states with complex values of the energy would decay under time evolution. Thus the spectrum obtained here has a part which is stable under time evolution

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and a decaying part corresponding to complex eigenvalues. This situation is similar to what happens in the case of many nuclei, where only a few energy states are stable and the others are resonances with characteristic decay widths. It would be interesting to find such physical applications of the model described here.

Acknowledgments

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